§9 PREQUANTIZATION

We now have collected all the ingredients which we need in order to continue the discussion in section 2 and describe the process of prequantization properly.

(9.1) <u>DEFINITION</u>: A symplectic manifold (M,ω) is said to be QUANTIZABLE if there exists a complex line bundle $L \to M$ with connection ∇ such that $\mathrm{Curv}(L,\nabla) = \omega$.
A PREQUANTUM BUNDLE on a symplectic manifold (M,ω) is a Hermitian line bundle (L,H) together with a a compatible connection such that $\mathrm{Curv}(L,\nabla) = \omega$.

Evidently, when (L,H,∇) is a prequantum bundle the base has to be quantizable. Conversely, on a given quantizable symplectic manifold there always exist prequantum bundles: Since the connection with $\mathrm{Curv}(L,\nabla) = \omega$ can be chosen to be real we can find (with the help of a partition of unity) a Hermitian metric H such that ∇ is compatible with respect to H (cf. section 7).

We have seen (see the end of section 6) that for a symplectic manifold (M,ω) the condition to be quantizable is a topological condition on M and ω:

The cohomology class induced by ω has to be an entire cohomology class (integrality condition). According to (6.5), (6.6) this is equivalent to [G1], i.e. to

$$\int_S \omega \in \mathbb{Z}$$

for all compact, oriented and closed surfaces $S \subset M$.

We come back to this condition later in the next section (§10) after we have given a short introduction into Čech cohomology in the appendix (§9A). We will construct a line bundle L with connection ∇ with $\mathrm{Curv}(L, \nabla) = \omega$ using the integrality condition; and we discuss the uniqueness of this construction.

As a result (M, ω) is quantizable if and only if $\int_S \omega \in \mathbb{Z}$ for all compact, oriented and closed surfaces $S \subset M$.

Before we come to these discussions we want to present examples and we describe the prequantization process.

(9.2) <u>EXAMPLES</u>: 1° Let $M = T^*Q$ be a cotangent bundle with $\omega = -d\lambda = dq^j \wedge dp_j$. The trivial bundle $L = M \times \mathbb{C}$ with the connection $\nabla_X f s_1 = (L_X f - 2\pi i \lambda(X) f) s_1$, $s_1(a) = (a, 1)$, has as its curvature

$$d(-\lambda) = \omega.$$

Since for every compact, oriented and closed surface $S \subset T^*Q$ one has

$$\int_S \omega = -\int_{\partial S} \lambda = 0$$

by Stokes's theorem $(\partial S = \emptyset)$, (T^*Q, ω) is quantizable.

2° In the same way a symplectic manifold (M, ω) for which ω is exact, i.e. $\omega = d\alpha$, $\alpha \in \Omega^1(M)$, is quantizable.

3° Let M be the two sphere $M = S^2$ of radius 1 with the symplectic form $\omega = c\,\mathrm{vol}$, $c \in \mathbb{R} \setminus \{0\}$, where vol is the standard volume form on S^2 ($\sin\theta\, d\theta \wedge d\varphi$ in polar coordinates). Since $\int_{S^2} \omega = c\,\mathrm{Vol}(S^2) = 4\pi c$ the symplectic manifold (S^2, ω) is quantizable in the sense of (9.1) (see §6 and §10) if and only if

$$4\pi c \in \mathbb{Z} \setminus \{0\},$$

i.e. $c = \frac{1}{4\pi} N$, $N \in \mathbb{Z} \setminus \{0\}$.

4° Hydrogen atom (Kepler problem):
The classical system is given by the manifold $M = T^*(\mathbb{R}^3 \setminus \{0\}) \cong (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ with the standard symplectic form $\omega = d(-\lambda) = \Sigma\, q^j \wedge p_j$ and Hamiltonian

$$H(q, p) = \frac{1}{2m} |p|^2 - \frac{k}{|q|},$$

where $m, k \in \mathbb{R}$, $m > 0$ and $k > 0$ $(k = e^2)$.

Since $dH \neq 0$, the "energy surface" for $E \in]-\infty, 0[$

$$\Sigma_E := H^{-1}(E) \subset M$$

is a 5-dimensional submanifold of M (hypersurface).
Identifying points $x, y \in \Sigma_E$ on a joint orbit leads to the orbit space

$$M_E = \Sigma_E / \sim ,$$

as a quotient, where the equivalence relation "$\sim$" is given by

$$x \sim y \;:\Longleftrightarrow\; \exists \text{ solution } \gamma \text{ of } \dot{\gamma} = X_H(\gamma) \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y.$$

It can be shown that the quotient M_E has the structure of a quotient manifold such that the projection $\pi_E : \Sigma_E \to M_E$ is an open map.
In concrete terms one finds a diffeomorphism

$$\Psi : M_E \longrightarrow S^2(mk) \times S^2(mk)$$

into the product of two 2 spheres of radius $r = mk$.
Ψ is given by $\Psi([a]) = (\rho I(a) + R(a), \rho I(a) - R(a))$, with $a = (q,p) \in \Sigma_E$ and where

$$I(a) = q \times p \qquad \text{"angular momentum"}$$

$$R(a) = I(a) \times p + mk \frac{q}{|q|} \qquad \text{"Runge Lenz vector"}$$

$$\rho = \sqrt{-2mE}$$

The standard symplectic form ω on $M = T^*(\mathbb{R}^3 \setminus \{0\})$ induces a 2 form on Σ_E by restriction which in turn induces a symplectic form ω_E on the orbit space M_E by $\pi_E^* \omega_E = \omega|_{\Sigma_E}$. The corresponding 2 form on $S^2(mk) \times S^2(mk)$ will be denoted by ω_E', hence,

$$\omega|_{\Sigma_E} = (\Psi \circ \pi_E)^* \omega_E' .$$

Quantizing the classical hydrogen atom at energy E is therefore equivalent to quantizing the symplectic manifold

$$(S^2(mk) \times S^2(mk), \omega_E').$$

It can be shown that ω_E' has the form

$$\omega_E' = \frac{1}{2\rho} \left(\frac{dx_1 \wedge dx_2}{x_3} + \frac{dy_1 \wedge dy_2}{y_3} \right), \quad x_3 \neq 0 \neq y_3 ,$$

where $(x_1, x_2, x_3) : S^2(mk) \hookrightarrow \mathbb{R}^3$ are the standard coordinates of $\mathbb{R}^3$ and similarly (y_1, y_2, y_3) for the second sphere $S^2(mk)$.

Because of $\int_{S^2(r)} r \frac{dx_1 \wedge dx_2}{x_3} = 4\pi (r)^2$ we obtain for $S := S^2(mk) \times \{y\} \subset S^2(mk) \times S^2(mk)$ the quantization condition

$$\int_S \omega_E' = \frac{1}{2\rho} \int_S \frac{dx_1 \wedge dx_2}{x_3} = \frac{4\pi}{2\rho} mk = N \in \mathbb{N} !$$

As a consequence, (M, ω_E) is quantizable only if $2\pi \frac{mk}{\rho} = N \in \mathbb{Z}$, i.e. if $4\pi^2 \frac{(mk)^2}{-2mE} = N^2$.

Hence, $-2mE = \frac{1}{N^2} 4\pi^2 m^2 k^2$, and we conclude that only the energy values

$$E_N = -\frac{2\pi^2}{N^2} m k^2$$

are quantizable (i.e. (M_{E_N}, ω_{E_N}) is quantizable). This corresponds to the BALMER sequence known from experimental physics!

(9.3) PROPOSITION: Let (L, ∇, H) be a prequantum line bundle over the symplectic manifold (M, ω). Then the map $q : \mathcal{E}(M, \mathbb{C}) \longrightarrow \mathrm{End}_{\mathbb{C}}(\Gamma(M, L))$

$$q(F) := -\frac{i}{2\pi} \nabla_{X_F} + F$$

satisfies the Dirac conditions

(D1) $q(1) = id_{\Gamma(M,L)}$

(D2) $[q(F), q(G)] = \frac{i}{2\pi} q(\{F, G\})$ for all $F, G \in \mathcal{E}(M, \mathbb{C})$.

□ Proof. The quantization condition is

$$\frac{1}{2\pi i}\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) = \mathrm{Curv}(\nabla, L)(X, Y) = \omega(X, Y).$$

Applied to $X = X_F$, $Y = Y_G$ and using $[X_F, X_G] = -X_{\{F,G\}}$ (1.10) this is

$$[\nabla_{X_F}, \nabla_{X_G}] = 2\pi i \{F, G\} - \nabla_{\{F,G\}}.$$

Hence,

$$[q(F),q(G)] = \left(\frac{i}{2\pi}\right)^2 \left(-\nabla_{\{F,G\}} + 2\pi i\{F,G\} - \frac{i}{2\pi}\left(L_{X_F}G + L_{X_G}F\right)\right)$$
$$= \frac{i}{2\pi}\left(-\frac{i}{2\pi}\nabla_{\{F,G\}} - \{F,G\}\right) + 2\frac{i}{2\pi}\{F,G\}$$
$$= \frac{i}{2\pi}\, q(\{F,G\}),$$

where we have used among others $L_{X_F}G = -\{F,G\}$. $\square$

(9.4) REMARK: One might not be content with the factor $\frac{i}{2\pi}$ in front of $q(\{F,G\})$ (in (D2)), preferring $\frac{1}{2\pi i}$ or $\frac{\hbar}{i}$ (where $\hbar = \frac{h}{2\pi}$ for some $h>0$) or even a general $c\in\mathbb{C}$. This can be achieved by changing the quantization condition $\mathrm{Curv}(L,\nabla) = \omega$ (cf. (9.1)) to a new condition $\mathrm{Curv}(L,\nabla) = -\omega$ or $\mathrm{Curv}(L,\nabla) = k\omega$ for a suitable $k\in\mathbb{C}$. The special outcome in (D2) in our case is due to the choices *

1) X_H defined by $i_{X_H}\omega = dH$ (and not $-dH$),
2) $\{f,g\} = \omega(X_f,X_g)$ (and not $-\omega(X_f,X_g)$)
3) $\mathrm{Curv}(L,\nabla) = \frac{1}{2\pi i}\left([\nabla_X,\nabla_Y] - \nabla_{[X,Y]}\right)$ (and not $= [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$)
4) $k=1$.

In general, to obtain an arbitrary c in (D2) we have in our conventions 1) – 3) to choose $k = \frac{i}{2\pi c}$ in the quantization condition 9.1: $\mathrm{Curv}(\nabla,L) = k\omega$. In particular, for $c = \frac{\hbar}{i} = \frac{h}{2\pi i}$ the quantization condition reads "$\ldots = -\frac{1}{h}\omega$".

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* cf. the "Sign Conventions"

In quantum models one usually wants to represent the observables on Hilbert spaces. It is not difficult to replace $\Gamma(M,L)$ by a natural Hilbert space of sections:

The symplectic form ω of a symplectic manifold (M,ω) always induces a volume form

$$\mathrm{vol} := (-1)^{\frac{1}{2}n(n-1)} \frac{1}{n!}\,\omega^n \in \Omega^{2n}(M),$$

where $\omega^n = \omega\wedge\omega\ldots\wedge\omega$ (n times), and $\dim M = 2n$. We obtain the Hilbert space $L^2(M,\mathrm{vol})$ by completing the prehilbert space

$$\{f : M \to \mathbb{C} \mid \int_M |f|^2 d\mathrm{vol} < \infty\}.$$

Now, let (L,H) be a Hermitian line bundle over a symplectic manifold (M,ω). Let

$$\{s \in \Gamma(M,L) \mid \int_M |s|^2 d\mathrm{vol} < \infty\} \subset \Gamma(M,L)$$

be the subspace of SQUARE INTEGRABLE smooth sections where $|s(a)|^2 = H(s(a),s(a))$, $a \in M$. This space is a pre hilbert space with respect to the scalar product

$$\langle s,t\rangle := \int_M H(s,t)\, d\mathrm{vol},$$

and its completion with respect to the norm

$$\|s\| := \sqrt[2]{\int_M |s|^2 ds}$$

is the Hilbert space $\mathcal{H} = \mathcal{H}(M,L)$ of square integrable sections.

(9.5) PROPOSITION: Let (M,ω) be a quantizable symplectic manifold and (L,∇,H) be a prequantum bundle. For every $F \in \mathcal{E}(M,\mathbb{R})$ for which $X_F \in \mathcal{V}(M)$ is a complete vector field on M the prequantum operator $q(F)$ is a well-defined self-adjoint operator

$$q(f) : \mathcal{H}(M,L) \longrightarrow \mathcal{H}(M,L) .$$

□ Proof: (Sketch) Let α be the connection form on $L^\times$, and let $V_F \in \mathcal{V}(L)$ be the vector field induced on L by X_F:

$$T\pi \circ V_F = X_F \circ \pi \quad \text{and} \quad \alpha(V_F) = F \circ \pi .$$

Let $\varphi : \mathbb{R}\times M \to M$ be the global flow on M satisfying

$$\dot{\varphi}(t,a) = [\varphi(t,a)]_a = X_F(a) \quad , \quad \varphi(0,a) = a .$$

(Such a flow exists globally, since X_F is complete.)
Let $\psi : \mathbb{R}\times L^\times \to L^\times$ the global flow of V_F
Now, on $\Gamma = \Gamma(M,L)$, we have a family

$$\rho_t : \Gamma \to \Gamma \; , \quad s \mapsto \psi_t^{-1} \circ s \circ \varphi_t \in \Gamma$$

of linear maps satisfying

$$\rho_t(\mathcal{H}\cap\Gamma) \subset \mathcal{H}$$

and

$$\frac{d}{dt}\rho_t(s)\Big|_{t=0} = \frac{i}{2\pi} q(F)(s)$$

Hence, $\rho_t(s) = \exp(-\frac{it}{2\pi} q(F))$, and $q(F)$ is s.a. according to Stone's thm. □

(9.6) EXAMPLE: Let us recall the example at the end of section 2, i.e. the case $M = T^*Q$, $Q \subset \mathbb{R}^n$ open. The example explains why we have to introduce and study polarizations (in section 11):

On M we regard the standard symplectic structure given by the 2 form $\omega = dq^j \wedge dp_j = d(-\lambda)$, $\lambda = p_j dq^j$, with respect to the canonical coordinates q^j, p_j on $T^*Q \subset \mathbb{R}^n \times \mathbb{R}^n$. Let $L = M \times \mathbb{C}$ the trivial line bundle with connection $\nabla_X f s_1 = (L_X f - 2\pi i \lambda(X) f) s_1$, $f \in \mathcal{E}(M)$, and with $s_1(a) := (q, 1)$ as before. Then $\mathrm{Curv}(\nabla, L) = \omega$. To determine the quantized $q(F) = \frac{1}{2\pi i} \nabla_{X_F} + F$ for $F = p_j$, $F = q^j$, we observe

$$\omega(X_{q^j}, Y) = dq^j(Y) \quad \text{by definition of } X_F \text{ and}$$

$$\omega(X_{q^j}, Y) = dq^k(X_{q^j}) dp_k(Y) - dq^k(Y) dp_k(X_{q^j}).$$

Consequently, $dq^k(X_{q^j}) = 0$ and $dp_k(X_{q^j}) = -\delta^j_k$. Therefore,

$$X_{q^j} = -\frac{\partial}{\partial p_j} \quad \text{and in the same way:} \quad X_{p_j} = \frac{\partial}{\partial q^j}.$$

Hence,

$$q(q^j) = -\frac{i}{2\pi} \nabla_{X_{q^j}} + q^j = -\frac{i}{2\pi}\left(-\frac{\partial}{\partial p_j} + 2\pi i \lambda\left(\frac{\partial}{\partial p_j}\right) q^j\right) + q^j,$$

$$q(q^j) = \frac{i}{2\pi}\frac{\partial}{\partial p_j} + q^j =: Q^j. \quad \text{Analogously,}$$

$$q(p_j) = -\frac{i}{2\pi}\frac{\partial}{\partial q^j} =: P_j.$$

As a result,

$$[q(q^j), q(p_k)] = [Q^j, P_k] = \frac{i}{2\pi}\delta^j_k = \frac{i}{2\pi} q(\{q^j, p_k\})$$

We see that the Dirac conditions are confirmed on the space $\Gamma(M,L) \cong \mathcal{E}(M)$ (which a consequence of proposition (9.3)), but also on the space

$$\mathcal{H} = \mathcal{H}(M,L) = L^2(M),$$

now for unbounded operators Q^j and P_k.
In comparison to the usual quantizations in this situation (and independently of what Hamiltonian $H \in \mathcal{E}(M)$ is considered) we observe that too many variables are involved: The wave function $\psi \in \mathcal{H}$ should depend on n variables and not on $2n$ variable

By replacing the Hilbert space by its subspace of all functions f of the form $f = g \circ \tau$, $g : Q \to \mathbb{C}$, for suitable g, e.g., we arrive at a function space with the correct dependencies and moreover

$$Q^j := q^j \quad \text{and} \quad P_j = -\frac{i}{2\pi}\frac{\partial}{\partial q^j}.$$

This is one of the possibilities of arriving at a correct space of wave functions using a suitable polarization as we will discuss in section 11.